Fibonacci mod k

In this section, we examine the question of which terms of the Fibonacci sequence have a given divisor k.

I start by giving out a table of the first 50 Fibonacci numbers—actually the first 51, because we begin with u_0=0 (and I have a reason for that which will soon become apparent).

n	u_n
0	0
1	1
2	1
3	2
4	3
5	5
6	8
7	13
8	21
9	34
10	55
11	89
12	144
13	233
14	377
15	610
16	987
17	1597
18	2584
19	4181
20	6765
21	10946
22	17711
23	28657
24	46368
25	75025

n	u_n
26	121393
27	196418
28	317811
29	514229
30	832040
31	1346269
32	2178309
33	3524578
34	5702887
35	9227465
36	14930352
37	24157817
38	39088169
39	63245986
40	102334155
41	165580141
42	267914296
43	433494437
44	701408733
45	1134903170
46	1836311903
47	2971215073
48	4807526976
49	7778742049
50	12586269025

Okay, which Fibonacci numbers are multiples of 10?
Well, that's easy enough—the multiples of 10 end in zero, and so we can read off u_{15}, u_{30}, and u_{45}. Okay, what's the next one? u_{60}! Well that's certainly a good guess. Can we be sure?

Well, we can check it out by continuing to generate the sequence from 50. That's not so hard to do when we realize that we need to keep track only of the last digit. And indeed we find that u_{60} is the next number to end in a zero.

Last digits	
n	u_n
49	9
50	5
51	4
52	9
53	3
54	2
55	5
56	7
57	2
58	9
59	1
60	0

So does the pattern continue?—u_{75} and then u_{90} etc.? That would certainly provide a real nice answer to my question:

But how can we prove that? Any ideas?

Well, yes! Look at u_{59}. It ends in a 1. And u_{60} ends in 0, so that means u_{61} will also end in a 1. So we have a consecutive 0, 1 and from that point on the sequence of last digits will continue as if from the beginning: 0,1,1,2,3,5,8,3…. We conclude:

the sequence of last digits is periodic with period 60.

So the next 0 will indeed be at u_{75} and then u_{90} etc. We conclude that for the entire infinite sequence:

the multiples of 10 occur every 15th number.

Well, that's a nice result. Now what I'm interested in here is the question of whether that kind of result might hold for other divisors as well. For example, suppose I took the divisor 11, and asked which Fibonacci numbers were divisible by 11.

The first multiple of 11 is clearly u_{10}. And from there, one easily sees that u_{20} u_{30} u_{40} and u_{50} are all multiples of 11. Okay, things are looking quite promising. Good.

But the big question here is whether the pattern goes on forever? Might there be some other multiples of 11 in the list. Like, say u_{36}?. Yep, those are the questions all right. I ask them and and I am met with silence from the class.

The divisor k=2.

Let's take a real simple divisor. Let's take k=2. Which terms are divisible by 2—that is, which terms are even?

It is clear from the table that, among the first 50 terms, every third term is even. Will it go on forever like this? Can we produce an argument that this pattern will hold indefinitely?

One obvious approach is to use what we've done for the last digits—because the last digit is enough to tell us whether a number is divisible by 2. We see the pattern holds for the first 60 numbers, so by the periodicity of the sequence of lasts digits, it will hold forever.

But there's a sense in which the "right" argument for the divisor 2 is much simpler than that. We shouldn't need so much of the table. The key here is to fasten attention on the sequence of "parities"—replace each even number by E and each odd number by O. We get

E O O E O O E O O E O O, etc

and we argue that it has to continue like this. The reason we can make this argument is that the parity of the *sum* of two numbers is determined by the parity of each summand: the sum of two odds or two evens is even, and the sum of an odd and an even is odd. So

The parity of the sum of two numbers is determined by the parity of the summands. This is an important argument to find, as it leads the way to mod- k arithmetic

the "divisor 2" behaviour is periodic, with period 3.

The divisor k=3.

Okay, let's move up. Take the divisor k=3. By looking at the sequence we can see that every 4th number is divisible by 3. Can we argue that this pattern must continue indefinitely

This is a great question to pose to the class, and then pick from the resulting forest(!) of hands, a volunteer to come to the board and deliver an explanation

Let's go for the parity argument from the case k=2. In this case, "even" ought to mean "multiple of three" and I guess there will have to be two kinds of "odd" numbers, odd-1 which would be 1 more than a multiple of three, and odd-2 which would be 2 more than a multiple of three. If we use 0,1 and 2 for these three "mod-3 parities," the pattern would go

Let's be careful with this—the crucial point is that, just like for the case k=2, the mod-3 parity of a sum can be deduced from the mod-3 parity of the summands—check it out.

0 1 1 2 0 2 2 1 0 1

There!—I stopped recording when I found a 0,1 because I knew that, from that point on, the pattern would just repeat. And since every fourth term is a zero, we conclude that every fourth Fibonacci number is divisible by 3 and:

the "divisor 3" behaviour is periodic, with period 8.

In more sophisticated mathematical language, we have shown that the Fibonacci sequence mod 3 is periodic with period 8.

Another way we might go with this is to try to imitate the solution for k=10 and think about what the last digits would be if we wrote the numbers in base 3. In fact, the last digits that would occur would be 0, 1 and 2, and we'd get exactly the pattern that we got above.

Other divisors.

Now we can do this for any divisor *k*—essentially what we record are the remainders that are left when multiples of *k* are removed, and the nice (and crucial!) property is that the sequence can be generated just using the Fibonacci rule on the remainders, as if they were the entire numbers. This is the general analogue of the observation that the parity of a sum is determined by the parity of the summands.

k	*Fibonacci sequence mod k*	*period*	*zeros/period*
2	0 1 1 0 1	3	1
3	0 1 1 2 0 1	4	1
4	0 1 1 2 3 1 0 1	6	1
5	0 1 1 2 3 0 3 3 1 4 0 4 4 3 2 0 2 2 4 1 0 1	20	4
6	0 1 1 2 3 5 2 1 3 4 1 5 0 5 5 4 3 1 4 5 3 2 5 1 0 1	24	2
7	0 1 1 2 3 5 1 6 0 6 6 5 4 2 6 1 0 1	16	2
8	0 1 1 2 3 5 0 5 5 2 7 1 0 1	12	2
9	0 1 1 2 3 5 8 4 3 7 1 8 0 8 8 7 6 4 1 5 6 2 8 1 0 1	24	2
10	0 1 1 2 3 5 8 3 1 4 5 9 4 3 7 0 7 7 4 1 5 6 1 7 8 5 3 8 1 9 0 9 9 8 7 5		
	2 7 9 6 5 1 6 7 3 0 3 3 6 9 5 4 9 3 2 5 7 2 9 1 0 1	60	4
11	0 1 1 2 3 5 8 2 10 1 0	10	1
12	0 1 1 2 3 5 8 1 9 10 7 5 0 5 5 10 3 1 4 5 9 2 11 1 0 1	24	2

In each case, we stop when we find a re-occurrence of the pair 0 1—from that point on, the sequence must repeat. Now the multiples of *k* are given by the zeros in the mod-*k* sequence, so we look at these. We note that there may be several in a period; in fact in the above examples, there are either 1, 2 or 4 zeros in a period. But one striking fact we observe is that

To generate these "mod-k" sequences, we add the two previous terms just as before, but then we "cast out k" if appropriate.

within a period the zeros are equally spaced

and so it follows that they will be equally spaced forever. To be specific, we see that:

The multiples of 2 occur every 3rd term
The multiples of 3 occur every 4th term
The multiples of 4 occur every 6th term
The multiples of 5 occur every 5th term
The multiples of 6 occur every 12th term

and so forth.

Now the question is—does this happen for every *k*? Take, for example, *k*=31. Is it the case that the multiples of 31 are equally spaced? Well, the first thing to ask is: how do we know there are *any* multiples of 31?

Could it be that no Fibonacci number is divisible by 31?

(other than 0 of course.) At first, it's not easy to see how to handle this question. But it turns out to have a very elegant analysis.

Start by thinking of what the Fibonacci sequence will look like mod 31:

0 1 1 2 3 5 8 13 21 3 24 27 20 16...

and so forth—on and on. Now what will happen? For example, will we get the periodic type of behaviour that we found for the smaller values of *k*? Let's fasten attention on that question—does the sequence have to repeat itself at some point? The answer is yes, and the reason is, essentially, that there are only a finite number of possible values for the sequence so something has to repeat. But it's important to be careful about this: for the sequence to repeat, what we need is a repetition of two consecutive values, so a proper argument has to consider the sequence of successive *pairs*:

Consider the sequence of pairs!—that's a nice piece of methodology and is in fact one of the key ideas behind the solution of second order recursive equations.

(0,1) (1,1) (1,2) (2,3) (3,5) (5,8) (8,13) (13,21) (21,3) (3,24)...

Now there are only a finite number of possible such pairs (31^2 to be precise, though one of these, (0,0), can't occur), so at some point, some *pair* must repeat. So from that point on (actually, from the previous occurrence of that pair) the sequence will be periodic.

But that doesn't show that the first repetition will occur with 0 1! In the above examples, that was always the case, but our argument doesn't yet show that. And *that's* the property that needs to hold in order for us to conclude that there's a second zero in the sequence.

So how might we argue this—that the first repetition has to be with the pair (0,1)? Well, what we have just argued is that there has to be some repeating pair—say, to take an specific example, (4,17). Then the sequence will have the form:

A note about the argument that the first repeat must be with 0 1. The orthodox mathematical way to make this argument is to suppose that 4 17 is the first repeat, and then get a contradiction. But my students seem to come up with the argument in the form at the right, and somehow this strikes me as more direct and more natural.

0 1 1 2.... *a* 4 17 21... *b* 4 17 21...

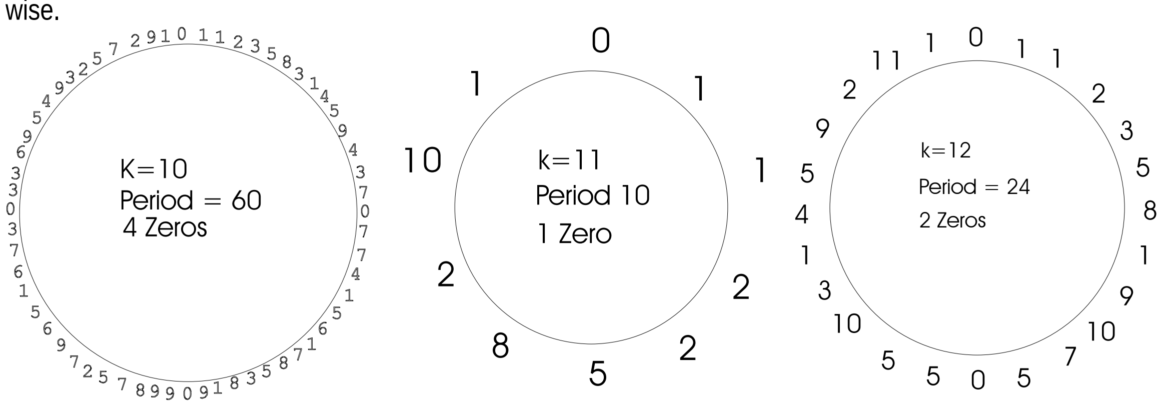
where I have used *a* and *b* to denote the terms immediately before the two instances of the repetition. But with two consecutive terms of the sequence, we can go backwards as well as forwards: we can deduce that *a* and *b* must both be equal to 13. *Thus the pair (13, 4) repeats already before (4, 17)*. We can continue backwards in this way until we reach 0 1. So the first repeat just has to be (0, 1). That's a nice argument.

We also observe, from counting the total number of possible pairs, that the repetition of (0,1) must occur within the first 31^2 terms. We state the general result:

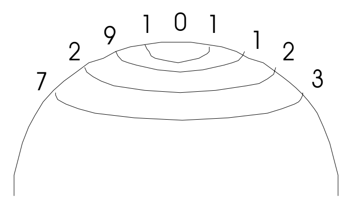
The Periodic Theorem.
The Fibonacci sequence mod k is periodic, with period less than k^2 .

Are we done? No!—we still haven't argued that the multiples of *k* are equally spaced. To get that, we have to know that the zeros that occur within a period are equally spaced. How are we to do that?

In thinking about this, it is useful to have some examples to study. Below I present the last three cases tabulated above: *k* = 10, 11 and 12—these illustrate the three possibilities for the number of zeros per period that we've seen so far. I celebrate the "cyclical" nature of a period by putting the terms in a circle, the primary zero at the top, with the sequence running clockwise.



Now study the pictures—what patterns can you see? The 0 at the top is always flanked by two 1's. As we move away from this 0, in both directions simultaneously, the pairs of numbers we get, one from the right and one from the left are either equal or they sum to *k* and this behaviour alternates.



Look at the *k*=10 case. The pairs on either side are (1, 1) then (9, 1), then (2, 2), then (7, 3), and so forth. And if you think about it, this behaviour follows easily from the Fibonacci rule!

Okay: let’s roll with this. Suppose we hit a 0 going in one direction. What will the corresponding term on the other side be? Well the pair of terms will be either equal or negatives. And *in both cases* the other term will be a 0 also! That is, as soon as you hit a 0 going in one direction, you have to hit one going the other way also. We conclude that the zero at the top must be half-way between the two zeros which flank it.

But this argument actually works starting at *any* zero, and working out in both directions, *every zero is half-way between the two zeros which flank it*. If you think about it, this means the zeros have to be equally spaced around the circle. And we have shown that the zeros are equally spaced within a period. And we are done!

This property of the pairs of numbers can be stated in a conceptually more powerful way: as we move away from zero, the pairs are alternately equal and negative *to one another mod k. For example, 9=-1 mod 10 because if you cast 10 out of 9 you’d get -1.. That’s the language we’ll use now: the pairs are alternately equal and negatives.*

Problems

1. Show that u_{2n} is a multiple of u_n . More generally, suppose n is a multiple of m. Does it follow that u_n is a multiple of u_m ?

2. TRUE or FALSE?

(a) If n divides into u_{250} and u_{150} then it must divide into u_{50}.
(b) u_{150} is divisible by both u_{15} and u_{10} .
(c) There is no Fibonacci number ending in the digits 63.
(d) There are infinitely many Fibonacci numbers ending in the digits 73.
(e) u_{116} is divisible by 1,542,687.
(f) There is at least one Fibonacci number with 27 digits.
(g) u_{100} and u_{103} have no common prime factor.

3. Find the greatest common divisor (GCD) of u_{200} and u_{309} .

4. TRUE or FALSE?

(a) If $a = n/m$ then $u_a = u_n/u_m$
(b) If $b = nm$ then $u_b = u_n u_m$
(c) If $c = \mathrm{LCM}(n,m)$ then $u_c = \mathrm{LCM}(u_n,u_m)$
(d) If $d = \mathrm{GCD}(n,m)$ then $u_d = \mathrm{GCD}(u_n,u_m)$
(e) u_{150} is divisible by the product of u_{15} and u_{10} .
(f) u_{180} is divisible by the product of u_{15} and u_{10} .

5. Investigate: u_n is prime if and only if n is prime.

6. If we use the Fibonacci rule but with a different starting point, then we get a different sequence. For example, if we start with 1, 3, we get the sequence

$$1, 3, 4, 7, 11, 18, 29, \ldots$$

This is known as the *Lucas' sequence.* Investigate the same divisibility questions for this sequence that we studied for the Fibonacci sequence. For example, given a divisor, are there always multiples of it in the sequence, and if so, are they equally spaced?

7. Can the product of two Fibonacci numbers ever be a Fibonacci number?

8. The sequence of ratios u_{2n}/ u_n starts off 1, 3, 4, 7, 11… That is, it seems to be the above Lucas' sequence. If you can show that it satisfies the Fibonacci sum rule, you will have a proof of this. Can you show that? [One idea is to try induction.]

9. There's one other interesting question, and that is what are the possible number of zeros that can occur inside a period of the Fibonacci sequence mod k? In the above examples, there were 1, 2 or 4 such zeros. Are there other possibilities, and what are they? The remarkable fact is that one of these possibilities always occurs—there are always either 1, 2, or 4 zeros in a period. Can you show this?

[I will get you started. Start at the top 0 and work out both ways till you come to the first zero on either side. Let these zeros look like $a\ 0\ a$ and $b\ 0\ b$. From our observation above, we must have $a=b$ or $a=-b$ (mod$k$). Argue that if $a=b$, then we either have the situation of $k=11$, with one zero in a period (if $a=1$) or the situation of $k=12$ with two zeros in a period (if $a\neq 1$). And if $a=-b$ (mod$k$)., we must have the situation of $k=10$ with four zeros in a period.]